1. Prolongation formula for general vector fields

Prolongation formula. Let $x \in (x^1, \dots, x^p)$ and $u = (u^1, \dots, u^q)$ and a vector field $V = \xi^i \frac{\partial}{\partial x^i} + \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha}$.[1] Then we have the following prolongation formula.

THEOREM 1.1. *For the vector field above,*

$$\mathsf{pr}^{(n)} V = V + \sum_{\alpha=1}^{q} \sum_{|J| \le n} \phi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}$$

where

$$\phi_\alpha^J = D_J(\phi_\alpha - \xi^i \cdot u_i^\alpha) + \xi^i \cdot u_{J,i}^\alpha \tag{1.1}$$

Furthermore for all J,

$$\phi_\alpha^{J,k} = D_k(\phi_\alpha^j) - (D_k \xi^i) u_{J,i}^\alpha. \tag{1.2}$$

REMARK 1.2. Recall that $\mathsf{pr}^{(n)} V$ is a vector field on $X \times U^{(n)}$ and defined as follows. Let $\exp(\varepsilon V) =: g_\varepsilon$ and $\mathsf{pr}^{(n)} g_\varepsilon(x, u^{(n)}) = (\tilde{x}(\varepsilon), \tilde{u}(\varepsilon))$, which is a curve parametrized by ε in $X \times U^{(n)}$. Now $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \mathsf{pr}^{(n)} g_\varepsilon(x, u^{(n)}) = \mathsf{pr}^{(n)} V(x, u^{(n)})$.

PROOF. Assume $n = 1$. Let $(\tilde{x}, \tilde{u}) = g_\varepsilon(x, u) := (\Xi_\varepsilon(x, u), \Phi_\varepsilon(x, u)$. Then $\frac{\partial\, \Xi}{\partial \varepsilon}\big|_{\varepsilon=0} = \xi(x, u)$, $\frac{\partial\, \Phi}{\partial \varepsilon}\big|_{\varepsilon=0} = \phi(x, u)$. Given $(x, u^{(1)}) \in X \times U^{(1)}$, let $f(x)$ be any function that fits this point i.e. $f^{(1)}(x) = u^{(1)}$. Then

$$\tilde{u} = \tilde{f}_\varepsilon(\tilde{x}) := (g_\varepsilon \cdot f)(\tilde{x}) = [\Phi_\varepsilon \circ (\mathbf{1} \times f)](x) = [\Phi_\varepsilon \circ (\mathbf{1} \times f)] \circ [\, \Xi_\varepsilon \circ (\mathbf{1} \times f)]^{-1}(\tilde{x}).$$

To get $\frac{\partial \tilde{u}^\alpha}{\partial \tilde{x}^k}$, we compute the Jacobian using the chain rule to have

$$[J\tilde{f}_\varepsilon](\tilde{x}) = J[\Phi_\varepsilon \circ (\mathbf{1} \times f)](x) \cdot J[\Xi_\varepsilon \circ (\mathbf{1} \times f)]^{-1}(\tilde{x})$$

whose (α, k) entry is $\tilde{u}_k^\alpha(\varepsilon)$. Differentiate in ε and evaluate the above at $\varepsilon = 0$ to find ϕ_α^k. Especially the right hand side becomes

$$J[\phi \circ (1 \times f)](x) \cdot I + Jf(x) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} J[\, \Xi_\varepsilon \circ (\mathbf{1} \times f)]^{-1}(\tilde{x})$$

which is $J[\phi \circ (\mathbf{1} \times f)](x) \cdot I - Jf(x) \cdot I \cdot J[\xi \circ (\mathbf{1} \times f)](x) \cdot I$ whose (α, k) entry is

$$\frac{\partial \phi^\alpha}{\partial x^k}(x, f(x)) - \frac{\partial f^\alpha}{\partial x^i} \cdot \frac{\partial \xi^i}{\partial x^k}(x, f(x)) = D_k \phi^\alpha - u_i^\alpha (D_k \xi^i).$$

That is,

$$\phi_\alpha^k := D_k \phi^\alpha - D_k \xi^i \cdot u_i^\alpha = D_k(\phi^\alpha - \xi^i u_i^\alpha) + \xi^i u_{ik}^\alpha$$

which is (1.2) for $n = 1$. We use induction. First note that $(n+1)$st jet $X \times U^{(n+1)}$ can be viewed as a subspace of $(X \times U^{(n)})^{(1)}$ as follows.

EXAMPLE 1.3. $p = 2$, $q = 1$. Then

$$\begin{aligned} X \times U^{(1)} &= \{(x, y, u, u_x, u_y)\} \\ X \times U^{(2)} &= \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\} \\ (X \times U^{(1)})^{(1)} &= \{(x, y, u, v, w, u_x, u_y, v_x, v_y, w_x, w_y)\} \end{aligned}$$

[1]We follow *Einstein* summation convention over *repeated indices.*

with $v = u_x$ and $w = u_y$. Regard $X \times U^{(2)}$ as a subset of $(X \times U^{(1)})^{(1)}$ defined by $u_x = v$, $u_y = w$ and $v_y = w_x$ etc.

We proceed on the induction on $|J|$. Note that $\mathsf{pr}^{(n-1)}V$ is a vector field on $X \times U^{(n-1)}$. For J with $|J| = n-1$

$$\phi_\alpha^{J,k} = D_k \phi_\alpha^J - D_k \xi^i \cdot u_{Ji}^\alpha$$

by the 1st prolongation formula, which still holds true for J with $|J| = n$. By induction hypothesis, this is in turn equal to

$$\begin{aligned} & D_k[D_j(\phi_\alpha - \xi^i u_i^\alpha) + \xi^i u_{Ji}^\alpha] - D_k \xi^i \cdot u_{Ji}^\alpha \\ = \; & D_k D_J(\phi_\alpha - \xi^i u_i^\alpha) + D_k \xi^i \cdot u_{Ji}^\alpha + \xi^i u_{Jik}^\alpha - D_k \xi^i \cdot u_{Ji}^\alpha \end{aligned}$$

fulfilling (1.1).

Example 1.4. Let $V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$ be infinitesimal rotation on $\{(x,u)\} = \mathbb{R}^2$. then

$$\begin{aligned} \mathsf{pr}^{(1)}V &= V + (1+u_x^2)\tfrac{\partial}{\partial x}, \\ \mathsf{pr}^{(2)}V &= \mathsf{pr}^{(1)}V + (3u_x \cdot u_{xx})\tfrac{\partial}{\partial u_{xx}} \end{aligned}$$

since $D_x(1+u_x^2) - \underbrace{D_x(-u)}_{=D_x\xi} \cdot u_{xx} = 2u_x u_{xx} + u_x u_{xx}$. Our differential equation is $u_{xx} = 0$ whose solutions are group of straight lines. Rotation of straight lines gives also straight lines hence we know our V is an infinitesimal symmetry. To prove it, it is enough to show $\mathsf{pr}^{(2)}V u_{xx} = 0$ on $u_{xx} = 0$. Actually

$$\begin{aligned} \mathsf{pr}^{(2)}V u_{xx} &= \left(-u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_{xx}} + (3u_x u_{xx})\frac{\partial}{\partial u_{xx}}\right) u_{xx} \\ &= 3u_x u_{xx} \end{aligned}$$

which is 0 on $u_{xx} = 0$.

Example 1.5. *Differential Invariant.* Given the graph of the function $u = f(x)$, $x \in \mathbb{R}$, its curvature $\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$ is rotation invariant. We remark here that κ is a function defined on $X \times U^{(2)}$, it is not V but $\mathsf{pr}^{(2)}V$ that is supposed to act on κ. Now $\mathsf{pr}^{(2)}V\left(\frac{u_{xx}}{(1+u_x^2)^{3/2}}\right) = \frac{3u_x u_{xx}(1+u_x^2)^{3/2} - u_{xx}(1+u_x^2)^{3/2}(1+u_x^2)^{1/2}u_x}{(1+u_x^2)^3} = 0$. Hence κ is a differential invariant of second order under rotation group.

Theorem 1.6. *Let $\Delta = (\Delta_1, \ldots, \Delta_l)$ be a system of differential equations defined on an open subset M of $\times U$. Then the set $\mathfrak{g}$ of all infinitesimal symmetries forms a Lie algebra. If $\mathfrak{g}$ is finite dimensional, the connected component of the symmetric group of $\Delta = 0$ is a local Lie group of transformations acting on M.* [2]

Proof. Let V, W be infinitesimal symmetries of $\Delta = 0$. Suppose that $\Delta = 0$ is of order n. In view of $\mathsf{pr}^{(n)}[V,W] = [\mathsf{pr}^{(n)}V, \mathsf{pr}^{(n)}W]$, we have $\mathsf{pr}^{(n)}[V,W]\Delta = [\mathsf{pr}^{(n)}V, \mathsf{pr}^{(n)}W]\Delta = \mathsf{pr}^{(n)}V(\mathsf{pr}^{(n)}W\Delta) - \mathsf{pr}^{(n)}W(\mathsf{pr}^{(n)}V\Delta)$ which vanishes on $\mathcal{S}_\Delta$ since $\mathsf{pr}^{(n)}V$, $\mathsf{pr}^{(n)}W$ are tangent to $\mathcal{S}_\Delta$ and $\mathsf{pr}^{(n)}V\Delta$ and $\mathsf{pr}^{(n)}W\Delta$ are 0 on $\mathcal{S}_\Delta$. Therefore $[V,W]$ is an infinitesimal symmetry.

[2]We only consider the Lie group of finite dimension.

2. Characteristic of Symmetries

For $V = \xi^i \frac{\partial}{\partial x^i} + \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$, let $Q_\alpha(x, u^{(1)}) := \phi_\alpha - \xi^i u_i^\alpha$, $\alpha = 1, \dots, q$. The q tuple $Q(x, u^{(1)}) = (Q_1, \dots, Q_q)$ is called *characteristic* of the vector field V. Then $\phi_\alpha^J = D_J Q_\alpha + \xi^i u_{Ji}^\alpha$ in (1.1) and

$$
\begin{aligned}
\mathsf{pr}^{(n)} V &= \sum_{i=1}^p \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{|J|\le n} \phi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha} \\
&= \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{|J|\le n} (D_J Q_\alpha + \xi^i u_{Ji}^\alpha)\frac{\partial}{\partial u_J^\alpha} \\
&= \sum_\alpha \sum_J D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{i=1}^p \xi^i \left(\frac{\partial}{\partial x^i} + u_{Ji}^\alpha \frac{\partial}{\partial u_J^\alpha}\right)
\end{aligned}
$$

Here we define $V_Q := \sum_{\alpha=1}^q Q_\alpha(x, u^{(1)})\frac{\partial}{\partial u^\alpha}$ and $\mathsf{pr}^{(n)} V_Q := \sum_{\alpha=1}^q \sum_{|J|\le n} D_J Q_\alpha \frac{\partial}{\partial u^\alpha}$. Noting $D_i = \frac{\partial}{\partial x^i} + u_{Ji}^\alpha \frac{\partial}{\partial u_J^\alpha}$, we have

$$
\mathsf{pr}^{(n)} V = \mathsf{pr}^{(n)} V_Q + \sum_{i=1}^p \xi^i D_i.
$$

EXERCISE 2.1. Complete a symmetric group for your choice differential equation.

3. Symmetric group of Heat equation

Let $p = n+1$, $q = 1$ and $u(x^1, \dots, x^p, t)$ be a $\mathcal{C}^2$ function defined on $\mathbb{R}^{n+1}$ that solves

$$
u_t = k\Delta u
$$

where $\Delta = \left(\frac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x^p}\right)^2$. This is called the heat conduction equation. For convenience sake, we assume $k = 1$ hereafter.

Physical motivation. Many physical laws are conservation laws and so is the heat equation. Let $\Omega \subset \mathbb{R}^n$ and $u(x,t)$ denote the temperature at $x \in \Omega$ and t. Then the vector $-\nabla_x u$ stands for the heat flux at (x,t). Total heat in Ω is $\int_\Omega u(x,t)dV(x)$, whose time derivative is the rate of heat increase in Ω. This should be caused by total heat flux into Ω across $\partial\Omega$. Hence

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u(x,t)dV(x) &= \int_{\partial\Omega} (-\nabla u)\cdot(-\vec{n})d\sigma \\
&= \int_\Omega \mathrm{div}\nabla u dV \\
&= \int_\Omega \Delta u dV.
\end{aligned}
$$

Since Ω was arbitrary, we deduce $u_t = \Delta u$.

3.1. Symmetric group of 1 dimensional heat equation. Let $u(x,y)$ be defined on $(x,t) \in \mathbb{R}^2$ that solves $u_t = u_{xx}$. We look for infinitesimal symmetry in the form $V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$. Let $\Delta(x,t,u,u_t,u_x,u_{xx},u_{xt},u_{tt}) := u_{xx} - u_t$. Then

$$
(\mathsf{pr}^{(2)} V)\Delta = 0 \text{ on } \Delta = 0
$$

is the equation to give the symmetry. Let $\mathsf{pr}^{(2)}V = V + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}$.